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# GLOBAL ATTRACTIVITY OF A RATIONAL DIFFERENCE EQUATION

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## ABSTRACT

In this paper we will discuss the global attractivity of difference equation

$$Z_{n+1} = \frac{a-\propto Z_{n-k}}{b+Z_n} \quad ,n = 0,1,2,\dots \tag{0.1}$$

Where $a \in (0,\infty), b \in (-\infty,0)\ , z_{-k}\ , z_{-k+1},\dots,z_0\ \in (-\infty\ ,0)$, $\alpha$ is non-negative real number and $k$ is a positive integer. For unique negative equilibrium point of equation (1.1), we will prove that it is a global attractor of equation with certain conditions.

**Keywords**: difference equation, global attractivity, stability

**Subject Classification** 39A10

## 1. INTRODUCTION

Difference equations are extensively studied over the years for its vast applications. Following is the glimpse of research done in the past years. In 2003 Owaidy et. al [1] investigated the global attractivity of sequence

$$x_{n+1} = \frac{-\alpha x_{n-1}}{\beta \pm x_n}$$

under specified conditions. They showed that these equations have zero equilibrium points which are global attractor .In 2003 Owaidy et. al [2] investigated local stability and boundedness character of positive solution of the difference equation

$$x_{n+1} = \alpha + \frac{{x_{n-1}}^p}{{x_n}^p}, n = 0,1,\dots$$

under the assumption $\alpha \in [0,\infty), p \in (1,\infty)$.

In 2003 Xing-Xue Yan et.al [3] investigated the global attractivity of the recursive sequence

$$x_{n+1} = \frac{\alpha - \beta x_n}{\gamma - x_{n-1}}, n = 0,1,\dots$$

where $\alpha \geq 0,\ \beta,\gamma > 0$ .They showed that the one positive equilibrium point of equation is a global attractor .
In 2004 Owaidy [4]investigated the global stability and periodicity of positive solution of difference equation

$$x_{n+1} = \alpha + \frac{x_{n-k}}{x_n}, n = 0,1,\dots$$

where $\alpha \in [0,\infty]$ and $x_{-1}, x_0$ are arbitrary chosen positive real numbers.

In 2004 Xing-Xue Yan et. al [5] investigated the behavior of all solutions of higher order nonlinear difference equation

$$x_{n+1}=\frac{a-bx_n}{A-\sum_{i=0}^{k}b_i x_{n-i}},n=0,1,...$$

In 2012 Mai Nam Phong [6] investigated the global attractivity of negative solutions of $(k+1)^{th}$ order rational difference equation

$$x_{n+1}=\frac{a-x_{n-k}}{b+x_{n-m}},n=0,1,...$$

where $a\in(0,\infty),b\in(-\infty,0),k,m$ are positive integers with $k\geq m$ and initial values $x_{-k},x_{-(k-1)},...,x_0\in(0,\infty)$.He showed that the unique negative equilibrium of above equation is a global attractor under certain conditions.

In 2018 Stephen & Muhammad Kalim [7] studied the global attractivity of a rational difference equation of order twenty $z_{n+1}=\alpha z_{n-9}+\frac{\beta z_{n-9}^2}{\gamma z_{n-9}+\delta z_{n-19}},n=0,1,2,...$ with the initial conditions $z_{-19},...,z_0$ are arbitrary positive real numbers and $\alpha,\beta,\gamma,\delta$ are constants. We obtained solutions and verified the obtained results by graphical examples.

In 2018 Stephen & Muhammad Kalim [8] studied the dynamic behavior of some higher order rational difference equations of the form $z_{n+1}=\frac{z_{n-20}}{\pm1\pm z_{n-6}z_{n-13}z_{n-20}},n=0,1,...$.where the initial conditions are arbitrary real numbers.To confirm the obtained solutions we considered some numerical examples by assigning different initial values with Matlab.Behavior of some more difference equations have been studied in [9-17]

Suppose that $\mathrm{I}$ is some interval of real numbers and $F$ a continuous function defined on $\mathrm{I}^{k+1}(k+1$ copies of $\mathrm{I}$ ),where $k$ is some natural number. Throughout this thesis, we consider the following difference equation

$$z_{n+1}=f(z_n,z_{n-1},...,z_{n-k}),n=0,1,... \tag{0.2}$$

For given initial values $z_{-k},z_{-(k-1)},...,z_0\in I$ .

### 1.1 Definition: (Invariant Interval)

An interval $L\subseteq I$ is invariant interval if $z_{-k},z_{-(k-1)},...,z_0\in L$ implies that $z_n\in L$ for all $n>0$.

### 1.2 Definition: (Equilibrium Point)

A point $\bar{z}\in I$ is called an equilibrium point of difference equation (2) if

$$\bar{z}=F(\bar{z},...,\bar{z}) \tag{0.3}$$

That is, $z_n=\bar{z}$ for $n\geq0$ is a solution of difference equation (1.2).

The linearized equation linked with the difference equation (1.2) with equilibrium point $\bar{z}$ is given by

$$y_{n+1}=\sum_{\alpha=0}^{k}\frac{\partial F(\bar{z},...,\bar{z})}{\partial z_\alpha}y_{n-\alpha} \tag{0.4}$$

and its characteristics equation is

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$$\mu^{k+1} = \sum_{\alpha=0}^{k} \frac{\partial F(\bar{z},\ldots,\bar{z})}{\partial z_\alpha} \mu_{k-\alpha} \tag{0.5}$$

### 1.3 Definition: (Periodicity)

A solution $\{z_n\}_{n=-k}^{\infty}$ of Equation (1.2) is called periodic with period $p$ if there exists an integer $p \geq 1$ such that $z_{n+p} = z_n$ for all $n \geq -k$ .If $z_{n+p} = z_n$ holds for smallest positive integer $p$ then solution $\{z_n\}_{n=-k}^{\infty}$ of Equation (1.2) is called periodic period of prime $p$ .

### 1.4 Definition:(Locally Stable)

If for every $\rho > 0$ there exist $\eta > 0$ such that for all $z_{-k}, z_{-(k-1)}, \ldots, z_0 \in I$ with $\sum_{\alpha=-k}^{0} \left|z_\alpha - \bar{z}\right| < \eta$ , we have $\left|z_n - \bar{z}\right| < \rho$ for all $n \geq -k$ .Then equilibrium point $\bar{z}$ of difference equation (1.2) is called locally stable.

### 1.5 Definition :(Locally Asymptotically Stable)

If equilibrium point $\bar{z}$ is locally stable, and there exist $\beta > 0$ such for all initial values $z_{-k}, z_{-(k-1)}, \ldots, z_0 \in I$ with $\sum_{\alpha=-k}^{0} \left|z_\alpha - \bar{z}\right| < \beta$ , we have, $\lim_{n\to\infty} z_n = \bar{z}$ .Then $\bar{z}$ of difference equation (1.2) is called locally asymptotically stable.

### 1.6 Definition :( Global Attractor)

If $z_{-k}, z_{-(k-1)}, \ldots, z_0 \in I$ always implies that $\lim_{n\to\infty} z_n = \bar{z}$ .Then $\bar{z}$ of difference equation (1.2) is called global attractor.

### 1.7 Definition :( Global Asymptotically Stable)

If $\bar{z}$ is locally asymptotically stable as well as a attractor .Then equilibrium point $\bar{z}$ of difference equation (1.2) is called global asymptotically stable.

### 1.8 Definition :( Unstable)

The equilibrium point $\bar{z}$ of difference equation is called unstable if it is not locally stable.

We also need the following theorems in the sequel.

**Theorem A**: Assume that $r, s \in R$ and $t \in \{0,1,2,\ldots.\}$ .Then $|r| + |s| < 1$ is a sufficient condition for asymptotic stability of the difference equation

$$z_{n+1} + r z_n + s z_{n-1} = 0 , \qquad n = 0,1,\ldots$$

**Theorem B**: Consider the difference equation

$$z_{n+1} = f(z_n, z_{n-k}), n = 0,1,... \quad (0.6)$$

Where $k \geq 1$ is an integer.Let $I = [p,q]$ be some interval of real numbers and assume $f:\ [p,q]\times[p,q]\to[p,q]$ is continuous function with properties:

(a) $f(a,b)$ is a decreasing function in $a$ and increasing function in $b$.

(b) If $(w,W)\in[p,q]\times[p,q]$ is a solution of $w = f(W,w)$ and $W = f(w,W)$ then $w = W$. Then equation (1.6) has a unique equilibrium $\bar{z}\in[p,q]$ and every solution of equation (1.6) converges to $\bar{z}$.

The following result, known as the Linearized Stability Theorem, is useful in determining the local stability character of the equilibrium point $\bar{z}$ of equation (1.2).

**Theorem C:** Consider the difference equation

$$z_{n+1} + a_k z_n + a_0 z_{n-k} = 0\ , \quad n = 0,1,...$$

Where $k \in \{1,2,...\}$ and $a_i$ real numbers for all $i$ .Then $\sum_{i=0}^{k} |a_i| < 1$ is a sufficient condition for the asymptotic stability of eq.(1.2).

For equilibrium point of equation

$$\bar{z} = \frac{-(b+\alpha) \pm \sqrt{(b+\alpha)^2 + 4a}}{2}$$

For linearized equation

$$f(s,v) = \frac{a - \alpha v}{b + s}$$

$$f_s^{'}(\bar{z},\bar{z}) = \frac{-(a - \alpha \bar{z})}{(b + \bar{z})^2} = -\frac{\bar{z}}{b + \bar{z}},\ f_v^{'}(\bar{z},\bar{z}) = -\frac{\alpha}{b + \bar{z}}$$

Linearized equation linked with eq (1.1) about equilibrium $\bar{z}$ is

$$P_{n+1} + \frac{\bar{z}}{b + \bar{z}} P_n + \frac{\alpha}{b + \bar{z}} P_{n-1} = 0$$

Its characteristic equation is

$$\lambda^{k+1} + \frac{\bar{z}}{b + \bar{z}} \lambda^k + \frac{\alpha}{b + \bar{z}} = 0$$

If $b < -\alpha$ then by definition negative equilibrium $\bar{z}$ of equation (1.1) is locally stable.

## 2. MAIN RESULTS

**Theorem 2.1**

Assume that $b < -\alpha$ then equation (1.1) has no negative prime period two solution.

**Proof**:

For purpose of contradiction ,assume there exist distinct negative real numbers $\phi$ and $\psi$ such that $..., \varphi, \psi, \varphi, \psi, ...$ is a prime period two solutions of eq (1.1).Then there are two cases

**Case (i):** If $k$ is odd then $z_{n+1} = z_{n-k}$ and $\varphi, \psi$ satisfy the system

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$$\begin{cases} \phi = \dfrac{a-\alpha\varphi}{b+\psi} \\ \psi = \dfrac{a-\alpha\psi}{b+\varphi} \end{cases} \Leftrightarrow \begin{cases} b\phi+\phi\psi = a-\alpha\varphi \\ b\psi+\phi\psi = a-\alpha\psi \end{cases}$$

On subtracting above equations we get

$$\Rightarrow (b+\alpha)(\phi-\psi)=0$$

In view $b+\alpha<0$ implies $\varphi=\psi$ which is contradiction that $\varphi\neq\psi$ .

**Case (ii)**: If $k$ is even then $z_n = z_{n-k} \neq z_{n+1}$ and $\varphi,\psi$ satisfy the system

$$\begin{cases} \phi = \dfrac{a-\alpha\psi}{b+\psi} \\ \psi = \dfrac{a-\alpha\phi}{b+\varphi} \end{cases} \Leftrightarrow \begin{cases} b\phi+\phi\psi = a-\alpha\psi \\ b\psi+\phi\psi = a-\alpha\phi \end{cases}$$

we get

$$\Rightarrow (b-\alpha)(\varphi-\psi)=0$$

In view $b-\alpha<0$ implies $\varphi=\psi$ which is contradiction that $\varphi\neq\psi$ .

**Theorem 2.2**

For $a>0$ ,Assume that $b\in\left(-\infty,\dfrac{-\alpha-\sqrt{\alpha^2+4a}}{2}\right]$ then $[b,0]$ is an invariant interval of eq(1.1).

**Proof:**
Set

$$f(s,v)=\frac{a-\alpha v}{b+s} \quad , \quad s,v\in(-\infty,0]$$

$$f_s' = \frac{-(a-\alpha v)}{(b+s)^2}<0,\ f_v' = -\frac{\alpha}{b+s}>0$$

Which implies that $f(s,v)$ is strictly decreasing function in $s$ and strictly increasing function in $v$ .Let $\{z_n\}$ be a solution of equation (1.1) along initial values $z_{-k}, z_{-k+1},\ldots,z_0\in[b,0]$ .For each fixed $s,v\in(-\infty,0]$ we obtain

$$z_1 = f(z_0,z_{-k}) < f(b,0) = \frac{a-\alpha(0)}{b+b} = \frac{a}{2b} < 0$$

$$z_1 = f(z_0,z_{-k}) > f(0,b) = \frac{a-\alpha b}{b} \geq b$$

From $b\leq\dfrac{-\alpha-\sqrt{\alpha^2+4a}}{2}$ we have $\dfrac{a-\alpha b}{b}\geq b$ .Hence $z_1\in[b,0]$ .Similarly

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$$z_2 = f(z_1, z_{1-k}) < f(b,0) = \frac{a-\alpha(0)}{b+b} = \frac{a}{2b} < 0$$

$$z_2 = f(z_1, z_{1-k}) > f(0,b) = \frac{a-\alpha b}{b} \geq b$$

This implies $z_2 \in [b,0]$.Similarly by induction $z_n \in [b,0]$ for $n \geq 1$ Hence $[b,0]$ is an invariant interval of equation (1.1).

**Theorem 2.3**

Assume that $b \in \left(-\infty, \frac{-\alpha-\sqrt{\alpha^2+4a}}{2}\right]$ the unique negative equilibrium $\bar{z}$ of equation (1.1) is a global attractor with a basin $T = [b,0]^{k+1}$.

**Proof:**
Consider the function

$$f(s,v) = \frac{a-\alpha v}{b+s}$$

Then $f:[b,0]\times[b,0] \to [b,0]$ is a continuous function and non-increasing function in $s$ and non-decreasing function in $v$ .Let $\{z_n\}$ be a solution of equation (1.1) with initial conditions $z_{-k}, z_{-k+1}, \ldots, z_0 \in T$ and $w, W \in [b,0]$ be a solution of system.

$$w = f(W,w), W = f(w,W)$$

On simplify above two systems we get $(b+\alpha)(w-W)=0$

Since $b+\alpha < 0 \Rightarrow w = W$ .By applying theorem (B) equation (1.1) has unique equilibrium point $\bar{z} \in [b,0]$ and every solution converges to $\bar{z}$ .So $\lim_{n\to\infty} z_n = \bar{z} \Rightarrow \bar{z}$ is a global attractor of equation (1.1).

## 3. CONCLUSION

we discussed the global attractivity of difference equation

$$z_{n+1} = \frac{a - \propto z_{n-k}}{b + z_n}, n = 0,1,2,\ldots$$

Where $a \in (0,\infty), b \in (-\infty,0)$ , $z_{-k}, z_{-k+1}, \ldots, z_0 \in (-\infty, 0)$, $\alpha$ is non-negative real number and $k$ is a positive integer. If $b < -\alpha$ then by definition negative equilibrium $\bar{z}$ of equation (1.1) is locally stable. Assume that $b < -\alpha$ then equation (1.1) has no negative prime period two solution. For $a > 0$ ,Assume that $b \in \left(-\infty, \frac{-\alpha-\sqrt{\alpha^2+4a}}{2}\right]$ then $[b,0]$ is an invariant interval of eq(1.1). Assume that

$b \in \left(-\infty, \frac{-\alpha-\sqrt{\alpha^2+4a}}{2}\right]$ the unique negative equilibrium $\bar{z}$ of equation (1.1) is a global attractor with a basin $T = [b,0]^{k+1}$.

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